The surface integral around $\sigma$ is evaluated only on the two surfaces of the hexahedron which lie in the $yz$ plane because $n_x = 0$ on all the other surfaces. By using the change of variables $\eta - y = s\epsilon$ and $\zeta - z = t\epsilon$, it can be shown that:

$$\lim_{\epsilon \to 0} \int_\sigma \psi_\xi (\xi - x,\ \eta - y,\ \zeta - z) n_x \mathrm{d}\sigma$$

$$= -2\int_{-\lambda}^{\lambda}\int_{-\lambda}^{\lambda} \frac{\mathrm{d}s\mathrm{d}t}{(1+s^2+t^2)^{3/2}}$$

$$= -8\arctan[\lambda^2(1+2\lambda^2)^{-1/2}] \tag{18}$$

If we differentiate Eq. (16) with respect to $x$ and use Eqs. (17 and 18), we obtain the integral equation:

$$u(x,y,z) = u_B(x,y,z) + \nu u^2(x,y,z)$$

$$+ \int_V K(\xi - x,\ \eta - y,\ \zeta - z) u^2(\xi,\eta,\zeta)\mathrm{d}V \tag{19}$$

where

$$u_B = \partial\phi_B / \partial x \tag{20a}$$

$$\nu = (1/\pi)\arctan[\lambda^2(1+2\lambda^2)^{-1/2}] \tag{20b}$$

$$K(\xi - x,\ \eta - y,\ \zeta - z) = \frac{(\eta - y)^2 + (\zeta - z)^2 - 2(\xi - x)^2}{8\pi[(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2]^{5/2}} \tag{20c}$$

The volume integral in Eq. (19) is defined according to Eq. (1) because the integrand is infinite at $(x,y,z)$. If $u$ is finite everywhere in the region $V$, then the use of spherical coordinates $\xi - x = r\sin\theta\cos\phi$, $\eta - y = r\sin\theta\ \sin\phi$, and $\zeta - z = r\ \cos\theta$ shows that $|Ku^2| < Mr^{-3}$ where $|u^2/4\pi| \leq M$. Hence, Eq. (2) is satisfied with $\mu = 3$. This indicates that if the volume integral in Eq. (19) converges, then it must be semiconvergent. The value of each of the last two terms in Eq. (19) depends on the shape of the cavity $\sigma$, hence on $\lambda$. But their sum is independent of $\lambda$, because it represents the derivative of a convergent integral.

The integral equation derived by Heaslet and Spreiter[1] can be obtained by taking the limit as $\lambda \to \infty$ to get $\nu = 1/2$. Surrounding the singularity by a vanishing sphere is equivalent to surrounding it by a vanishing cube.[6] The resulting integral equation is deduced by choosing $\lambda = 1$ so that $\nu = 1/6$.

There arises a situation similar to what occurs for the two-dimensional flow[2]; namely, by varying the value of $\lambda$, it appears that we are getting different integral equations. In reality, the integral equations are the same provided we define the volume integral in Eq. (19) according to Eq. (1) and use the same cavity around which Eq. (18) is evaluated.

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# Overexpanded Two-Dimensional Transonic Free Jet

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## Nomenclature

| | |
|---|---|
| $G$ | = Green's function |
| $H$ | = nozzle exit half height |
| $i$ | = $\sqrt{-1}$ |
| $K$ | = transonic similarity parameter |
| $\ell$ | = integration path |
| $M$ | = Mach number |
| $m$ | = index |
| $n$ | = denotes normal to boundary |
| $p$ | = pressure |
| $t$ | = dummy variable |
| $U$ | = velocity |
| $X, Y, y$ | = Cartesian coordinates |
| $x$ | = transonic coordinate = $X/\sqrt{\epsilon}H$ |
| $\gamma$ | = ratio of specific heats |
| $\epsilon$ | = perturbation parameter = $(U_e - U_\infty)/U_\infty$ |
| $\xi, \eta$ | = dummy variables |
| $\Phi$ | = exact potential |
| $\phi$ | = perturbation potential |

*Subscripts*

| | |
|---|---|
| $e$ | = nozzle exit plane |
| $s$ | = denotes shock wave surface |
| $\infty$ | = conditions on jet boundaries |

*Superscript*

| | |
|---|---|
| $*$ | = locally sonic conditions |

## I. Introduction

COMPLICATED flow patterns develop when an overexpanded transonic nozzle jet flow adjusts to a high-back pressure. The shock wave patterns that form depend on the exit pressure, Mach number, and far-field pressure. In general, two basic regimes can be distinguished. The first involves supersonic flow on the boundaries and the flow adjustment occurs through either regular reflection of an oblique shock wave or through Mach reflection. In both cases, regions of embedded subsonic flow can occur.[1] For exit Mach numbers less than 1.486 ($\gamma = 1.4$), it is not certain[2] that Mach reflection occurs for two-dimensional flows.

When the pressure ratio $p_e/p_\infty$ is sufficiently high, subsonic flow occurs on the jet boundaries and a strong curved shock stands at the nozzle exit. The flowfield bounded by the curved shock waves, jet boundaries, and downstream infinity is entirely subsonic and, hence, elliptic. In contrast to jet flows with supersonic boundaries, calculations of the flowfield with subsonic boundaries must be carried out to downstream infinity, at least in the inviscid approximation.

In the transonic regime, shock waves can be considered isentropic as entropy changes are $0[(M_e^2 - 1)^3]$ and for exit Mach numbers sufficiently close to one, a small disturbance approximation to the full-potential equation developed. For subsonic boundaries, a far-field approximation for the flow at

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infinity can be derived and numerical calculations performed in a finite region. In the following, equations appropriate for an overexpanded two-dimensional transonic ($M_e > 1$) jet with subsonic or supersonic boundaries are formulated. A numerical method for solution of the boundary value problem is briefly described and typical numerical results presented.

## II. Analysis

The transonic approximations are herein applied to an overexpanded free jet issuing from a two-dimensional nozzle. The flow is uniform at $X=0$, $|Y| \leq H$ (see Fig. 1) and the exit Mach number $M_e$ is greater than 1. The pressure on the jet boundaries can prescribe either a supersonic flow with reflected oblique shock wave or a strong shock in the nozzle exit plane. In the latter case, the flow at downstream infinity is uniform with pressure $p_\infty$ and velocity $U_\infty$.

Since changes in Mach number or normalized velocity are small, a perturbation parameter

$$\epsilon = (U_e - U_\infty)/U_\infty \tag{1}$$

may be defined and the potential expanded as

$$\Phi = U_\infty \{ X + \epsilon^{3/2} H \phi(x,y) + \ldots \} \tag{2}$$

where $y$ is the vertical coordinate normalized by $H$ and $x$ is a stretched coordinate defined as $X/H\sqrt{\epsilon}$. The local pressure and velocity components in terms of the disturbance potential are:

$$\Phi_X / U_\infty = 1 + \epsilon\phi_x + \ldots, \quad \Phi_Y/U_\infty = \epsilon^{3/2}\phi_y + \ldots \tag{3a}$$

$$p/p_\infty = 1 - \epsilon\gamma M_\infty^2 \phi_x + \ldots \tag{3b}$$

and the perturbation parameter related to $p_e/p_\infty$, $M_\infty$ is:

$$\epsilon = \frac{1 - p_e/p_\infty}{\gamma M_\infty^2}$$

To first order, the equations of motion reduce to the transonic small-disturbance equation:

$$(K - (\gamma+1)\phi_x)\phi_{xx} + \phi_{yy} = 0 \tag{4}$$

where the transonic similarity parameter is defined as:

$$K = (1 - M_\infty^2)/\epsilon \tag{5}$$

The shock jump relations, obtained from integration of the divergence form of Eq. (4), and the requirement that there is no jump in tangential velocity across the shock wave are:

$$\left[K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right]\mathrm{d}y_s - [\phi_y]\mathrm{d}x_s = 0 \tag{6a}$$

$$[\phi_x]\mathrm{d}x_s + [\phi_y]\mathrm{d}y_s = 0 \tag{6b}$$

A unique solution for the overexpanded jet is specified with the following boundary conditions:

$$\phi_x = 1, \ \phi_y = 0 \qquad x=0, \ |y| \leq 1 \tag{7a}$$

$$\phi_x = 0 \qquad x > 0, \ y = \pm 1 \tag{7b}$$

$$\phi, \ \phi_x, \ \phi_y \to 0 \qquad x \to +\infty \tag{7c}$$

The first condition represents a uniform flow with velocity $U_e$, and $\phi(0,y)$ can be taken to be zero. The second boundary condition specifies a constant pressure on the jet boundary and since $[\phi(0, \pm 1)] = 0$ (no jump in tangential velocity across the shock wave), the disturbance potential is also zero on this boundary. Actually, the jet boundary is more precisely described by:

$$y = \pm 1 + \epsilon^2 G(x), \qquad G'(x) = \phi_y(x, \pm 1) \tag{8}$$

However, as $\epsilon \sim 0$, the boundary condition can be applied on $y = \pm 1$. The final boundary condition is applicable for flow with subsonic boundaries and represents the decay of disturbances at downstream infinity.

The behavior of the disturbance potential as $x \to +\infty$ can be studied in more detail in terms of a far-field solution. Considering Eq. (4) as an elliptic equation with a nonlinear right-hand side

$$L\phi = K\phi_{xx} + \phi_{yy} = \frac{\gamma+1}{2}(\phi_x^2)_x$$

A Green's formula can be constructed wherein

$$\phi(x,y) = \int_{-1}^{1}\int_0^\infty GL\phi \,\mathrm{d}\xi\,\mathrm{d}\eta + \oint\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right)\mathrm{d}\ell \tag{9}$$

and $G$ is the Green's function satisfying

$$KG_{\xi\xi} + G_{\eta\eta} = \delta(x+\xi)\delta(y-\eta) - \delta(x-\xi)\delta(y-\eta) \tag{10}$$

with homogeneous Dirichlet conditions on all boundaries. Equation (9) can be reduced, using the boundary conditions for $\phi$ and $G$ and the shock jump conditions [Eqs. (6)], to the following form:

$$\phi(x,y) = -\frac{\gamma+1}{2}\int_{-1}^{1}\int_0^\infty G_\xi \phi_\xi^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \tag{11}$$

$G(\xi,\eta;x,y)$ is determined by the inverse of the Fourier transform ($\tilde{G}$) of Eq. (10):

$$\tilde{G}(t,\eta;x,y) = -\sqrt{\frac{2}{\pi}}\frac{\sinh itx}{t\sqrt{K}\sinh 2\sqrt{K}t}\sinh\sqrt{K}(1 \mp y)t \cdot \sinh\sqrt{K}(1 \pm \eta)t \quad \begin{cases} -1 < \eta < y \\ y < \eta < 1 \end{cases} \tag{12}$$

The inverse Fourier transform is:

$$G(\xi,\eta;x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\xi}\sinh itx}{\sqrt{K}t\sinh 2\sqrt{k}t} \cdot \sinh\sqrt{k}(1 \mp y)t \cdot \sinh\sqrt{K}(1 \pm \eta)t\,\mathrm{d}t \tag{13}$$

and the integal can be evaluated by complex contour integration in the upper and lower-half planes. There are simple poles at $m\pi i/2\sqrt{K}$, $m>0$ when $\xi > x$, $\xi + x > 0$, $\eta + y \leq 2$ and similarly simple poles at $-m\pi i/2\sqrt{K}$, $m > 0$ for $\xi < x$, $\xi + x < 0$, $\eta + y \leq 2$. Accordingly, for $0 \leq \xi < x$, the Green's function is given by:

$$G(\xi,\eta;x,y) = \frac{2}{\pi\sqrt{K}}\sum_{m=1}^{\infty}\frac{(-)^{m+1}}{m}e^{-m\pi x/2\sqrt{K}} \cdot \sinh\frac{m\pi\xi}{2\sqrt{K}}\sin\frac{m\pi}{2}(1 \mp y)\sin\frac{m\pi}{2}(1 \pm \eta) \tag{14}$$

and, further, for $x \gg \xi$, one term in the series is sufficient and

$$G_\xi = \frac{1}{K}e^{-\pi x/2\sqrt{K}}\cos\frac{\pi}{2}\eta\cos\frac{\pi}{2}y \tag{15}$$

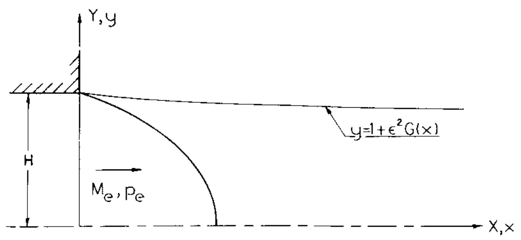


**Fig. 1 Geometry.**

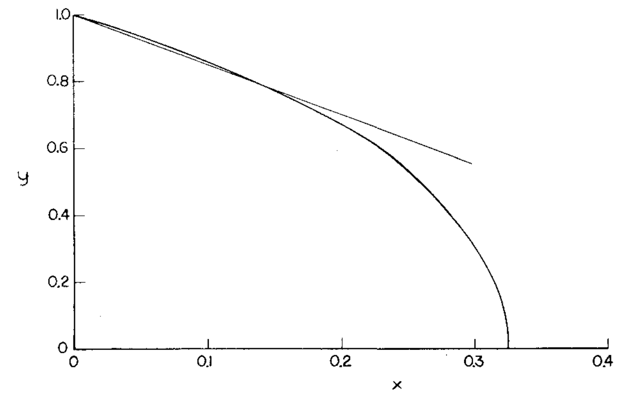


**Fig. 2 Shock wave shape; $M_e = 1.1, p_e/p_\infty = 0.85$.**

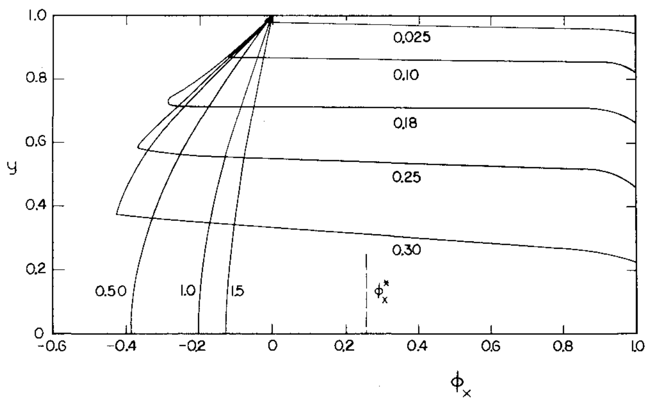


**Fig. 3 Velocity profiles at stations ($x$ = const) downstream of nozzle exit; $M_e = 1.1, p_e/p_\infty = 0.85$.**

The disturbance potential in the far field can, therefore, be represented by:

$$\phi(x,y) = -\frac{\gamma+1}{K}\cos\frac{\pi}{2}y e^{-\pi x/2\sqrt{K}}\int_0^1\int_0^\infty \cos\frac{\pi}{2}\eta\phi_\xi^2 d\xi d\eta \quad (16)$$

and exhibits an exponential decay.

## III. Results

The boundary value problem posed by Eqs. (4) and (7) is solved using a type-dependent relaxation scheme[3] in which elliptic points are overrelaxed and hyperbolic points underrelaxed. For subsonic flows on the boundaries, the boundary condition at infinity [Eq. (7c)] was approximated by Eq. (16) and a converged solution obtained iteratively through periodic readjustment of the source strength, as represented by the double integral in Eq. (16).

Results of a calculation for a uniform overexpanded jet with an initial Mach number of $M_e = 1.1$ and pressure ratio $p_e/p_\infty = 0.85$ ($\epsilon = 0.115$, K = 0.622) are presented in Figs. 2 and 3. For this calculation, $x = 2$ was sufficiently far from the nozzle exit for application of the far-field boundary condition. In Fig. 2, the boundary of the uniform supersonic region is shown, as well as the continuation of the oblique shock wave which forms at the corner. The shock point operator[4] tends to smear the shock wave over several (four) mesh points; therefore, the shock position is somewhat on the downstream side of the boundary curve. The effective shock "thickness" as well as the decay in velocity can be observed in the near-field velocity profiles shown in Fig. 3. A very fine mesh was used for $x < 0.5$ and, similarly, near the jet boundary $y \sim 1$.

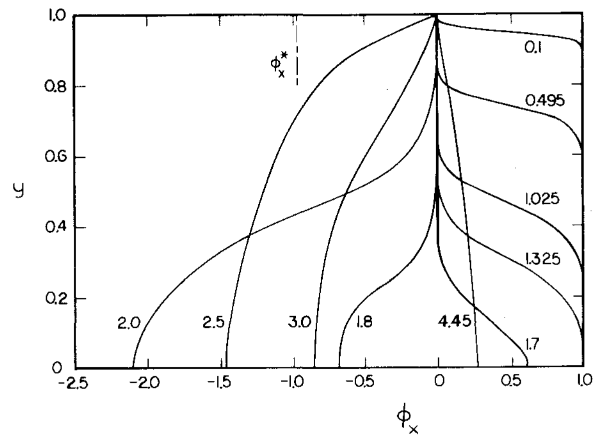


**Fig. 4 Velocity profiles at stations ($x$ = const) downstream of nozzle exit; $M_e = 1.1, p_e/p_\infty = 0.937$.**

Velocity profiles calculated for an overexpanded jet with supersonic flow on the boundaries ($M_e = 1.1$, $p_e/p_\infty = 0.937$, $\epsilon = 0.041$, $K = -2.33$) are shown in Fig. 4. The calculation was carried out to $x = 5$. Apparent in the velocity profiles is the region of embedded subsonic flow which occurs downstream of the reflected shock wave.

## IV. Discussion

The solution procedure just outlined for overexpanded transonic nozzle flows is applicable for flows with subsonic or supersonic boundaries. The restriction of uniform flow at the nozzle exit can be removed without difficulty,[5] although the flow in the exit plane must remain supersonic. This study is currently being extended to axisymmetric jet flows. Surprisingly, experimental results are lacking, at least for two-dimensional exit flows, that give shock wave shapes and near-field velocity fields.


## Acknowledgment

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